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EQUILIBRIUM STABILITY IN OLIGOPOLISTIC MARKETS: A TOPOLOGICAL ANALYSIS OF COURNOT AND STACKELBERG MODELS[1]

OLİGOPOLİSTİK PİYASALARDA DENGE KARARLILIĞI: COURNOT VE STACKELBERG MODELLERİNİN TOPOLOJİK ANALİZİ

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ARTICLE INFO	ABSTRACT
Received 10.02.2026 **Revized** 20.03.2026 **Accepted** 28.03.2026 **Article Classification:** Research Article **JEL Codes** C02 C72 D43	In this study, the existence of Nash equilibrium in oligopolistic market structures is investigated using topological fixed-point theorems, moving beyond classical computational methods such as differential calculus and first-order optimization conditions. The Cournot (simultaneous move) and Stackelberg (leader-follower) models are examined, analyzing the compactness and convexity of strategy spaces alongside the continuity of reaction functions. The existence of Nash equilibrium is proven via the Brouwer and Kakutani Fixed Point Theorems without relying on computational methods. Furthermore, comparative static analysis (comparison of equilibrium states) and welfare analysis between the models are presented. The topological proofs are supported by Monte Carlo simulations under stochastic cost shocks, demonstrating that while the Cournot model is structurally more stable, the Stackelberg model is more stable, the Stackelberg model is more vulnerable to crises. However, the findings reveal a fundamental stability-efficiency trade-off regarding economic welfare. **Keywords:** Game Theory, Nash Equilibrium, Brouwer Fixed Point Theorem, Kakutani Fixed Point Theorem, Cournot Model, Stackelberg Model, Oligopoly, Stochastic Analysis, Welfare Analysis.

MAKALE BİLGİSİ	ÖZ
Gönderilme Tarihi 10.02.2026 **Revizyon Tarihi** 20.03.2026 **Kabul Tarihi** 28.03.2026 **Makale Kategorisi** Araştırma Makalesi **JEL Kodları** C02 C72 D43	Bu çalışmada, oligopolistik piyasa yapılarında Nash dengesinin varlığı, diferansiyel hesap ve birinci derece optimizasyon koşulları gibi klasik hesaplama yöntemlerinin ötesinde topolojik sabit nokta teoremleri kullanılarak incelenmiştir. Cournot (eş anlı karar) ve Stackelberg (lider-takipçi) modelleri ele alınmış, strateji uzaylarının kompaktlığı ve konveksliği ile tepki fonksiyonlarının sürekliliği analiz edilmiştir. Nash dengesinin varlığı, hesaplamalı yöntemlere gerek duyulmaksızın Brouwer ve Kakutani Sabit Nokta Teoremleri aracılığıyla ispatlanmıştır. Ayrıca modeller arası karşılaştırmalı statik analiz (denge durumlarının kıyaslanması) ve refah analizi sunulmuştur. Topolojik ispatlar, stokastik maliyet şokları altında Monte Carlo simülasyonlarıyla desteklenmiş ve Cournot modelinin yapısal olarak daha kararlı, Stackelberg modelinin ise daha kriz kırılganı olduğu, ancak ekonomik refah açısından bir kararlılık-verimlilik ödünleşimi barındırdığı ispatlanmıştır. **Anahtar Kelimeler:** Oyun Teorisi, Nash Dengesi, Brouwer Sabit Nokta Teoremi, Kakutani Sabit Nokta Teoremi, Cournot Modeli, Stackelberg Modeli, Oligopoli, Stokastik Analiz, Refah Analizi

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Introduction

Situations where economic decision-makers cannot act independently, and the payoff of one player depends on the strategies of other players, are defined as strategic interactions in microeconomics. Game theory is the discipline that mathematically models these interactions under the framework of "games." Formulated by John Nash (1950), the Nash equilibrium represents a state where players maximize their payoffs by taking the strategies of others as given, and no player has an incentive to unilaterally deviate from their chosen strategy (Nash, 1950: p. 48).

Oligopolistic markets are market structures characterized by a small number of sellers, where the decisions made by one firm directly affect the profits of its competitors. In this study, two fundamental strategic models based on quantity competition are examined: The Cournot Model (Cournot, 1838), where firms determine their output levels simultaneously, and the Stackelberg Model, where decisions are made sequentially within a leader-follower hierarchy. In both approaches, the analysis is constructed upon the reaction functions firms develop against their competitors' strategies within the framework of profit maximization problems. These two fundamental models are frequently subjected to comparative analysis in the recent literature regarding profit and production volumes under variables such as asymmetric costs and the number of competitors (Gao, 2024).

In engineering and economics literature, particularly in studies concerning energy markets and production planning (Pemberton & Rau, 2015), the analysis of equilibrium points is generally confined to differential calculus methods and first-order conditions (FOC), where optimal points are determined by taking the derivatives of profit functions. For instance, in the works of Varian (1992) and Tirole (1988), equilibrium analysis is predominantly conducted through these algebraic methods. However, from a mathematical analysis perspective, before proceeding to the numerical solution of a system of equations, it must be theoretically proven that the solution set is non-empty (proof of existence). The existence of an equilibrium point depends on the topological structure of the strategy space and the continuity of the defined functions, independent of computational methods.

The objective of this study is to prove the existence of Nash equilibrium in oligopolistic markets via topological fixed-point theorems and to statistically test the structural stability of these equilibrium points under exogenous shocks. In the first stage, the compactness and convexity of the strategy spaces and the continuity of the reaction functions are analyzed, and the existence of equilibrium is demonstrated using the Brouwer and Kakutani fixed-point theorems.

In the second stage, the economic efficiency and resistance to cost shocks of these topologically proven equilibrium points are examined. Exogenous cost shocks affecting the market are integrated into the model as random variables and a stochastic variance analysis is conducted using Monte Carlo simulation (10.000 iterations). Based on the analysis, a fundamental "Stability-Efficiency Trade-off" for oligopolistic markets is introduced to the literature. The findings reveal that the Cournot model is structurally more stable in absorbing exogenous shocks, whereas the Stackelberg model, despite generating higher welfare due to the first-mover advantage, is statistically more fragile against crises.

1. Preliminaries

This section presents the fundamental topological concepts and fixed-point theorems utilized to solve equilibrium existence problems in game theory. The standard Euclidean space $\mathbb{R}^n$ is taken as the reference for all definitions.

Theorem 2.1. (Heine-Borel Theorem) A subset $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bouded. That is, every open cover of the set K has a finite subcover:

For $\forall \{U_\alpha\}_{\alpha \epsilon I}$ open cover, $\exists\ \alpha_1, \dots, \alpha_k$ such that $K \subset \bigcup_{j=1}^{k} U_{\alpha j}$

Theorem 2.2. (Tychonoff's Theorem) The cartesian product of any (finite or infinite) family of compact spaces is compact. If $\{X_\alpha\}_{\alpha \epsilon I}$ is a family of compact spaces, then $\prod_{\alpha \epsilon I} X_\alpha$ is compact with respect to the product topology.

Definition 2.1. (Upper Hemicontinuity) Let X and Y be two metric spaces, and let $\mathcal{P}(Y)$ denote the power set of Y. Given a set-valued correspondence $\Phi: X \longrightarrow \mathcal{P}(Y)$, Φ is said to be **upper hemicontinuous** if for every point $x \in X$ and for every open set $V \subset Y$ satisfying $\Phi(\mathrm{x}) \subset V$, there exists an open neighborhood U of x such that $\Phi(\mathrm{z}) \subset \mathrm{V}$ for all $z \in U$.

Theorem 2.3. (Berge's Maximum Theorem) Let X and Y be topological spaces, $f: X \times Y \longrightarrow \mathbb{R}$ be a continuous function, and $\Gamma: X \to \mathcal{P}(Y)$ be a continuous correspondence with compact values. Define the maximization problem for $x \in X$:

$$M(x) = \max\{f(x,y) \mid y \in \Gamma(x)\}$$
$$\Phi(x) = \{y \in \Gamma(x) \mid f(x,y) = M(x)\}$$

Under these conditions, the value function $M(x)$ is continuous, and the optimal choice corresponce $\Phi(x)$ is upper hemicontinuous.

Theorem 2.4. (Brouwer Fixed-Point Theorem) Let $\mathrm{K} \subset \mathbb{R}^{\mathrm{m}}$ be a non-empty, compact, and convex set. If $\mathrm{f}: \mathrm{K} \to \mathrm{K}$ is a continuous function, then there exists at least one fixed point $\mathrm{x} \in \mathrm{K}$ such that $\mathrm{f(x)} = \mathrm{x}$ (Ok, 2007).

Theorem 2.4. (Kakutani Fixed-Point Theorem) Let $K \subset \mathbb{R}^{\mathrm{m}}$ be a non-empty, compact, and convex set. If the set-valued correspondence $\Phi: \mathrm{K} \to \mathcal{P}(\mathrm{K})$ is upper hemicontinuous and $\Phi(\mathrm{x})$ is non-empty, convex, and closed for all $\mathrm{x} \in \mathrm{K}$, then there exists a fixed point such that $\mathrm{x} \in \Phi(\mathrm{x})$ (Kakutani, 1941).

2. Topological Analysis and the Existence of Equilibrium

The fundamental space over which the model is defined is the metric space $(\mathbb{R}^{\mathrm{m}}, d)$ equipped with the standard Euclidean metric $d(x,y) = ||x-y||_2$ and the topological space $(\mathbb{R}^{\mathrm{m}}, \tau)$ with topology τ.

For each firm $i \in \{1,2\}$ the strategy space is defined as follows, where $K > 0$ represents the maximum production capacity:

$$S_i = [0, K] \subset \mathbb{R}$$

$S_i \subset \mathbb{R}$ is closed and bounded $\Rightarrow S_i$ is compact by the Heine-Borel Theorem:

$$\forall x, y \in S_i,\ \forall \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in S_i \Rightarrow S_i \text{ is convex.}$$

The joint strategy space is expressed as the Cartesian product of the strategy sets of both firms:

$$S = S_1 \times S_2 = \{(q_1, q_2) \in \mathbb{R}^2 \mid 0 \le q_1 \le K,\ 0 \le q_2 \le K\}$$

By Tychonoff's Theorem, $\prod S_i$ is compact $\Rightarrow S$ is a compact and convex metric space.

2.1. General Case: Quasi-Concave Profit Functions and Kakutani's Proof
Assumption 1: $\Pi_i : S \to \mathbb{R}$ is a continuous function, $\Pi_i \in C^0(S)$.
Assumption 2: $\forall q_{-i} \in S_{-i}$, the function $q_i \mapsto \Pi_i(q_i, q_{-i})$ is quasi-concave.

Firm i's optimal choice correspondence is defined as:

$$\Phi_i : S_{-i} \to \mathcal{P}(S_i), \quad \Phi_i(q_{-i}) \coloneqq \arg\max_{q_i \in S_i} \Pi_i(q_i, q_{-i})$$

The topological conditions required for the Kakutani Fixed-Point Theorem are analyzed as follows:

Non-emptiness and Compactness of the Image Set:
$\Pi_i \in C^0$ and S_i is compact $\Rightarrow \max \Pi_i$ exists by Weierstrass's Theorem.

$$\therefore \forall q_{-i} \in S_{-i}, \quad \Phi_i(q_{-i}) \neq \emptyset \quad \text{and} \quad \Phi_i(q_{-i}) \text{ is compact.}$$

Convexity of the Image Set:
By Assumption 2, Π_i is quasi-concave $\Rightarrow \forall \alpha \in \mathbb{R}$, the upper contour set is convex:

$$U_\alpha(q_{-i}) = \{q_i \in S_i \mid \Pi_i(q_i, q_{-i}) \ge \alpha\}$$

$$\Phi_i(q_{-i}) = U_{max} \Rightarrow \Phi_i(q_{-i}) \text{ is convex.}$$

Upper Hemicontinuity:
Define a constant, compact-valued, and continuous correspondence:

$$\Gamma : S_{-i} \to \mathcal{P}(S_i), \quad \Gamma(q_{-i}) = S_i$$

$\Pi_i \in C^0$ and Γ is continuous $\Rightarrow \Phi_i$ is upper hemicontinuous by Berge's Maximum Theorem.

Proof: Let the joint best-response correspondence be defined as:

$$\Phi : S \to \mathcal{P}(S), \quad \Phi(q) \coloneqq \Phi_1(q_2) \times \Phi_2(q_1)$$

S is compact and convex. Φ is upper hemicontinuous, and for all $q \in S, \Phi(q) \subset S$ is a non-empty, compact, and convex set. The compactness and convexity of the strategy space ensure the existence of equilibrium by guaranteeing that the solution set is bounded and without holes; upper hemicontinuity ensures that minor changes in competitors' strategies do not lead to sudden and massive jumps in the optimal response set. According to the Kakutani Fixed-Point Theorem:

$$\exists q^* \in S \quad \text{such that} \quad q^* \in \Phi(q^*)$$

q^* is the Nash equilibrium of the system. ■

2.2. Special Case: Strictly Concave Profit Functions and Brouwer's Proof

Assumption 3: $\forall q_{-i} \in S_{-i}$, the function Π_i is strictly concave with respect to q_i.

$$\Pi_i \in C^2 \Rightarrow \frac{\partial^2 \Pi_i}{\partial q_i^2} < 0$$

The evolution of the conditions in Kakutani's proof under strict concavity is as follows:

Uniqueness:

Π_i is strictly concave, and (S_i, d) is compact and convex $\Rightarrow$ the global maximum point is unique.

$$|\Phi_i(q_{-i})| = 1 \Rightarrow \text{The correspondence } \Phi_i \text{ reduces to a single-valued function:}$$

$$\Phi_i(q_{-i}) \equiv \{R_i(q_{-i})\}, \quad R_i : S_{-i} \to S_i$$

Continuity:

The correspondence Φ_i is upper hemicontinuous and single-valued ⇒ topological continuity condition in metric spaces is satisfied.

$$\therefore R_i \in C^0(S_{-i}, S_i)$$

Proof: Let the continuous vector transformation be defined as:

$$F: S \to S, \quad F(q_1, q_2) \coloneqq \big(R_1(q_2), R_2(q_1)\big)$$

Since (S, d) is a compact and convex topological space and $F \in C^0(S)$, according to the Brouwer Fixed-Point Theorem:

$$\exists q^* \in S \quad \text{such that} \quad F(q^*) = q^*$$

q^* is the Nash equilibrium under strict concavity. ■

3. Mathematical Solution of the Linear Model (Application)

In this section, a linear demand function is selected as a concrete application of the general theory proven in previous sections. For the function $P(Q) = a - bQ$, since $P'(Q) = -b < 0$ and

$P''(Q) = 0 \leq 0$, it perfectly satisfies the concavity and monotonicity assumptions detailed in Section 2.1.

The inverse market demand function is defined linearly as follows:

$$P(Q) = a - bQ = a - b(q_1 + q_2) \tag{1}$$

Here, $a > 0$ represents the reservation price, and $b > 0$ represents the slope of the demand. It is assumed that both firms have a constant marginal cost equal to $c(a > c)$. Accordingly, the cost function of firm i is $C_i(q_i) = c \cdot q_i$.

3.1. Cournot (Simultaneous) Equilibrium Solution

Each firm maximizes its profit by taking its competitor's production quantity as given. The profit function for Firm 1 is:

$$\Pi_1 = P(Q) \cdot q_1 - C(q_1) = \big(a - b(q_1 + q_2)\big)q_1 - cq_1 \tag{2}$$

To find the first-order condition, taking the partial derivative with respect to q_1 and setting it to zero yields:

$$\frac{\partial \Pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c = 0 \Rightarrow q_1 = \frac{a - c - bq_2}{2b} \tag{3}$$

Under the symmetric firm assumption ($q_1 = q_2 = q^*$), the general form of the Cournot Nash equilibrium is obtained as:

$$q^* = \frac{a - c}{3b} \tag{4}$$

3.2. Stackelberg (Leader-Follower) Equilibrium Solution

In this model, Firm 1 (Leader) moves by knowing Firm 2's (Follower) reaction function $\left(R_2(q_1) = \frac{a-c-bq_1}{2b}\right)$ and incorporates it as a constraint into its own profit function. (Fudenberg & Tirole, 1991) The leader's maximization problem is:

$$\max_{q_1} \Pi_1 = \Big(a - b\big(q_1 + R_2(q_1)\big)\Big) q_1 - cq_1 \tag{5}$$

Substituting the follower's reaction function:

$$\Pi_1 = \left(a - b\left(q_1 + \frac{a - c - bq_1}{2b}\right)\right) q_1 - cq_1 \tag{6}$$

Simplifying the expression:

$$\Pi_1 = \left(\frac{a - c}{2b}\right) q_1 - \frac{b}{2} q_1^2 \tag{7}$$

Applying the first-order condition:

$$\frac{d\Pi_1}{dq_1} = \frac{a - c}{2} - bq_1 = 0 \Rightarrow q_1^* = \frac{a - c}{2b} \tag{8}$$

The follower's output is found by substituting the leader's production quantity into the follower's reaction function:

$$q_2^* = \frac{a - c - b\left(\frac{a-c}{2b}\right)}{2b} = \frac{a-c}{4b} \tag{9}$$

3.3. Theoretical Formulation of the Stochastic Model (Asymmetric Costs)
The symmetric and deterministic cost assumption may be insufficient to reflect exogenous shocks in the market. Recent studies, such as Gao (2024), emphasize that deterministic models with symmetric costs are insufficient to capture market realities and that introducing asymmetric production costs significantly alters firms' optimal output levels and profit distributions. Building upon this premise, the marginal costs of the firms are redefined as independent and normally distributed continuous random variables:

$$c_1, c_2 \sim \mathcal{N}(\mu, \sigma^2) \tag{10}$$

Here, μ represents the expected marginal cost, and σ^2 represents the market uncertainty (the variance of cost shocks). When the first-order optimization conditions (FOC) are re-derived under asymmetric costs (c1≠ c2), the production quantities for the stochastic Cournot Nash equilibrium are obtained as follows:

$$q_{1,C}^* = \frac{a - 2c_1 - c_2}{3b}, \qquad q_{2,C}^* = \frac{a - 2c_2 + c_1}{3b} \tag{11}$$

In the Stackelberg model where Firm 1 is the leader, the Stochastic Stackelberg equilibrium production quantities are calculated as below, as a result of the Leader maximizing its profit by taking the follower's asymmetric reaction function ($q_2 = \frac{a-c_2-bq_1}{2b}$) as a constraint:

$$q_{1,S}^* = \frac{a - 2c_1 - c_2}{2b}, \qquad q_{2,S}^* = \frac{a - 2c_1 + 3c_2}{4b} \tag{12}$$

This asymmetric formulation will form the basis of a dynamic analysis and simulation that assumes the symmetric deterministic model as the expected value.

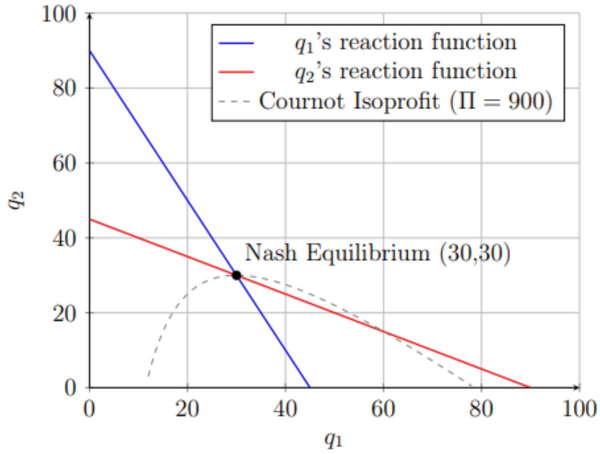
4. Numerical Application and Findings
To test the validity of the theoretical model, widely used parameters in literature were selected: $a = 100, b = 1, c = 10$. Accordingly, the demand function is determined as $P = 100 - Q$ and the cost function as $C = 10q$.

4.1 Cournot Equilibrium Calculation
The firms' reaction functions are calculated as $q_i = 45 - 0.5q_j$. Solving the system of equations yields:

- **Firm Outputs:** $q_1 = 30, q_2 = 30$ units.
- **Total Output:** $Q = 60$ units.
- **Market Price:** $P = 100 - 60 = 40$ currency units.
- **Firms Profits:** $\Pi = 900$ currency units for each firm.

This result indicates that the point (30, 30) is a stable Nash equilibrium in the strategy space.

Figure 1: *Geometric Representation of Nash Equilibrium in the Cournot Model*



4.2 Stackelberg Equilibrium (Subgame Perfect Nash Equilibrium) Calculation

Since the game is dynamic in the Stackelberg model, the equilibrium is found via **backward induction** to obtain the Subgame Perfect Nash Equilibrium (SPNE). The procedure is executed from the end to the beginning as follows:

Stage 2: Follower's Decision (End of the Game)

The follower firm (Firm 2) maximizes its profit after observing the leader's (Firm 1) production quantity (q_1):

$$\underset{q_2}{max}\,\Pi_2 = (100 - q_1 - q_2)q_2 - 10q_2$$

Applying the first-order optimization condition, the follower's reaction function (R_2) is found:

$$\frac{\partial \Pi_2}{\partial q_2} = 90 - q_1 - 2q_2 = 0 \rightarrow q_2 = 45 - 0.5q_1$$

Stage 1: Leader's Decision (Beginning of the Game)

The leader knows this rational response of the follower in advance and substitutes it as a constraint into its own profit function:

$$\max_{q_1} \Pi_1 = \big(100 - q_1 - (45 - 0.5q_1)\big)q_1 - 10q_1$$

When rearranged, the leader's profit function is reduced to a single variable:

$$\Pi_1 = (45 - 0.5q_1)q_1 = 45q_1 - 0.5q_1^2$$

Taking the derivative to find the output that maximizes the leader's profit:

$$\frac{d\Pi_1}{dq_1} = 45 - q_1 = 0 \rightarrow q_1^* = 45$$

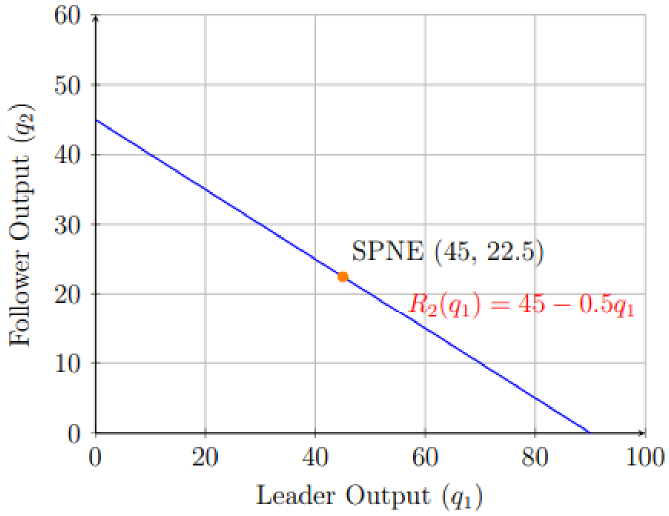
This value is substituted into the follower's reaction function to find the SPNE point:

$$q_2^* = 45 - 0.5(45) = 22.5$$

Equilibrium Results:

- **Leader Output** (q_1^*): 45 units.
- **Follower Output** (q_2^*): 22.5 units.
- **Market Price:** $P = 100 - (45 + 22.5) = 32.5$ currency units.
- **Profits:** Leader $\Pi_1 = 1012.5$, Follower $\Pi_2 = 506.25$.

Figure 2: *Location of the Subgame Perfect Equilibrium on the Follower's Reaction Function in the Stackelberg Model*



5. Welfare Analysis

In this section, the effects of the Cournot and Stackelberg equilibria on social welfare are analyzed through **Consumer Surplus (CS)** and **Producer Surplus (PS)** (Dixit, 1986). As established in the foundational oligopoly literature (Vives, 1999), the sequential nature of the Stackelberg model typically yields a higher aggregate output and lower market price than the simultaneous Cournot model, theoretically leading to a higher total welfare. Total Welfare (TW) is calculated as the sum of these two values ($TW = CS + PS$).

5.1. Cournot Model Welfare Analysis

In the symmetric Cournot equilibrium, the price is calculated as $P_c = 40$ and total quantity as $Q_c = 60$.

- **Consumer Surplus:** Consumer surplus is the area between the demand curve and the price line:

$$CS_c = \frac{1}{2}(a - P_c)Q_c = \frac{1}{2}(100 - 40) \cdot 60 = \frac{1}{2}(60 \cdot 60) = 1800$$

- **Producer Surplus**: Equal to the total profit of the producers:

$$PS_c = \Pi_1 + \Pi_2 = 900 + 900 = 1800$$

- **Total Welfare:**

$$TW_c = CS_c + PS_c = 1800 + 1800 = 3600$$

5.2. Stackelberg Model Welfare Analysis
In the Stackelberg equilibrium, the price is calculated as $P_s = 32.5$ and total quantity as $Q_s = 67.5$.

- **Consumer Surplus:** Due to increased production and decreased price, consumer surplus has expanded:

$$Cs_s = \frac{1}{2}(a - P_s)Q_s = \frac{1}{2}(100 - 32.5) \cdot 67.5 = \frac{1}{2}(67.5 \cdot 67.5) = 2278.125$$

- **Producer Surplus:** The sum of the Leader's and Follower's profits:

$$PS_s = \Pi_L + \Pi_T = 1012.5 + 506.25 = 1518.75$$

- **Total Welfare:**

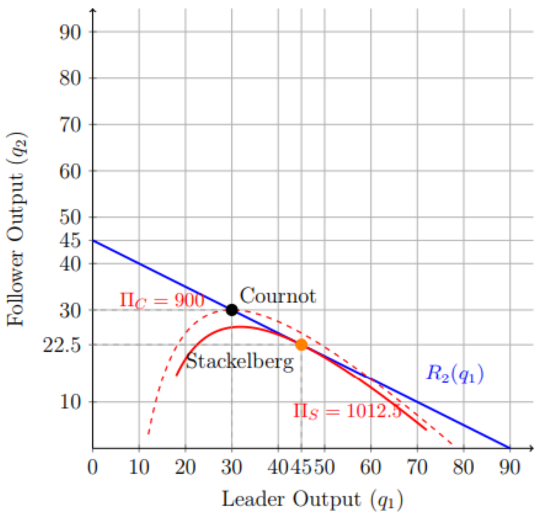
$$TW_s = 2278.125 + 1518.75 = 3796.875$$

6. Isoprofit Curves and the Geometry of Stackelberg Equilibrium
In this section, the difference between the Cournot and Stackelberg equilibria is analyzed geometrically via the isoprofit curves of the firms.
Definition (Isoprofit Curve): It is the curve formed by all (q_1, q_2) combinations that yield a firm a constant profit level $\overline{\Pi}$. The isoprofit equation for the leader firm (Firm 1) is:

$$\overline{\Pi} = \big(100 - (q_1 + q_2)\big)q_1 - 10q_1 \rightarrow q_2 = 90 - q_1 - \frac{\overline{\Pi}_1}{q_1} \tag{13}$$

These curves represent higher profit levels as they move closer to the q1 axis (Pemberton & Rau, 2015: p. 112).
6.1 Comparison of Equilibria
In the graph below, the follower firm's reaction function (blue line) and the leader firm's isoprofit curves for different profit levels (red curves) are depicted.

Figure 3: *Geometric Representation of Stackelberg Equilibrium: The leader firm's highest profit curve (Π_S) is tangent to the Follower's reaction line (R_2). The Cournot equilibrium (Π_C) is an intersection point at a lower profit level.*



7. Equilibrium Stability Under Stochastic Shocks and Variance Analysis

The structural stability of topologically proven Nash equilibrium points in oligopolistic markets against exogenous market shocks is analytically investigated. The asymmetric cost shocks encountered in the market were integrated into the model in Section 3.3 under the assumption that firms' marginal costs are independent, normally distributed continuous random variables: $c_1, c_2 \sim \mathcal{N}(\mu, \sigma^2)$. Here, μ expresses the expected average cost, and σ^2 signifies the severity of exogenous shocks.

7.1. Expected Value (Mean) Calculations

Using the linearity property of the expected value operator ($\mathbb{E}$), the long-term average behaviors of the production quantities are calculated. Since the expected value of costs is $\mathbb{E}[c_1] = \mathbb{E}[c_2] = \mu$, the expected values of the production quantities directly converge to the symmetric deterministic model results.

Cournot Model Expected Output:

$$\mathbb{E}[q_1, c] = \mathbb{E}\left[\frac{a - 2c_1 + c_2}{3b}\right] = \frac{a - 2\mu + \mu}{3b} = \frac{a - \mu}{3b} \tag{14}$$

Stackelberg Model Expected Output (Leader Firm):

$$\mathbb{E}[q_1, S] = \mathbb{E}\left[\frac{a - 2c_1 + c_2}{2b}\right] = \frac{a - 2\mu + \mu}{2b} = \frac{a - \mu}{2b} \tag{15}$$

7.2. Analytical Variance (Stability) Calculations

The stability of the system under exogenous shocks is analyzed using the variance (Var)operator, which measures the amount of deviation in the strategy space. Utilizing the property $Var(aX \pm bY) = a^2Var(X) + b^2Var(Y)$ for independent random variables X and Y, the analytical variance bounds of both models are derived.

Cournot Model Output Variance:

$$Var(q_1, C) = Var\left(\frac{a - 2c_1 + c_2}{3b}\right) = \left(-\frac{2}{3b}\right)^2 Var(c_1) + \left(\frac{1}{3b}\right)^2 Var(c_2) = \frac{5\sigma^2}{9b^2}$$

Stackelberg Model Leader Firm Variance (q_1):

$$Var(q_1, S) = Var\left(\frac{a - 2c_1 + c_2}{2b}\right) = \left(-\frac{2}{2b}\right)^2 Var(c_1) + \left(\frac{1}{2b}\right)^2 Var(c_2) = \frac{5\sigma^2}{4b^2}$$

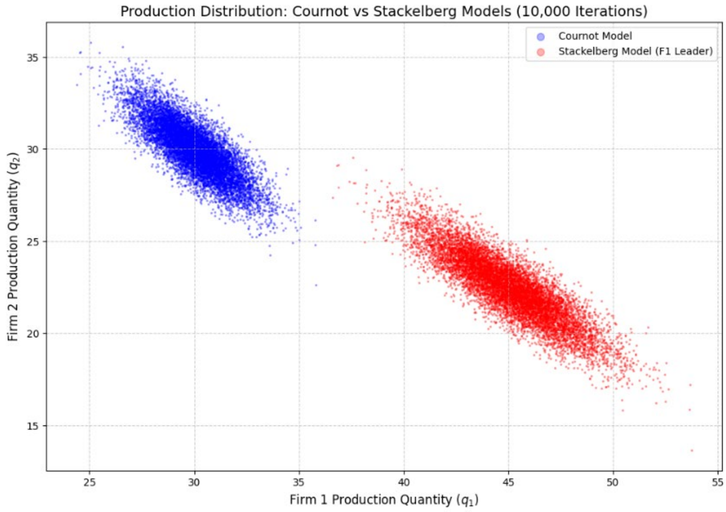
Stackelberg Model Follower Firm Variance (q_2):

$$Var(q_2, S) = Var\left(\frac{a - 2c_1 + 3c_2}{4b}\right) = \frac{13\sigma^2}{16b^2}$$

7.3. Numerical Application and Interpretation of Statistical Results

The theoretically obtained variance bounds were calculated under the assumptions of a demand slope $b = 1$ and shock severity $\sigma = 2 \Rightarrow \sigma^2 = 4$, and compared with the empirical data obtained from a Monte Carlo simulation of 10,000 iterations. The scatter plot generated in the Python environment is presented in Figure 4. Furthermore, while recent comparative studies (e.g., Gao, 2024) rely solely on static tabular data for deterministic outcomes and highlight the lack of visual data representation as a
major limitation, this study directly addresses that methodological gap by visualizing the stochastic dispersion of equilibria through scatter plots.

Figure 4: *Dispersion of Cournot and Stackelberg Equilibrium Points in the Strategy Space under Stochastic Shocks (10,000 Iterations)*



The simulation results perfectly verified the theoretical limits:

- Cournot Variance: Theoretical value $\frac{5.4}{9}$ ≈ 2.22 (Simulation: 2.25)
- Stackelberg Leader Variance: Theoretical value $\frac{5.4}{4}$ = 5.00 (Simulation: 5.06)
- Stackelberg FollowerVariance: Theoretical value $\frac{13.4}{16}$ = 3.25 (Simulation: 3.28)

When subjected to the same exogenous cost shocks (c_1, c_2), the dispersion of Stackelberg equilibrium points is significantly higher than that of Cournot points (5.00 >2.22). Mathematically, this proves that the strategic uncertainty created by firms making simultaneous decisions in the Cournot model serves to absorb exogenous shocks entering the system. In the Stackelberg model, the leader firm adding the follower's reaction function to its optimization problem as a strict constraint makes the system more fragile to parametric changes.

7.4 Expected Total Welfare Analysis

To measure the overall efficiency of the models on the market, the Total Welfare $(TW = CS + PS)$ function is examined under stochastic shocks. Since consumer surplus is a quadratic function $\left(CS = \frac{1}{2}bQ^2\right)$, the welfare of expected values and the expected value of welfare are not equal due to Jensen's Inequality $(TW(\mathbb{E}[Q]) \neq \mathbb{E}[TW(Q)])$. Thus, expected total welfare is calculated and integrated individually for each random shock realization.

According to the 10.000-iteration simulation results, the expected total welfare realized as **3604,02 for the Cournot model** and **3802,79 for the Stackelberg model**.

8. Conclusion and Evaluation

In this study, the equilibrium analysis of Cournot and Stackelberg oligopoly models was proven using Brouwer and Kakutani fixed-point theorems in topological spaces, advancing beyond traditional differential calculus methods. After proving the existence of the fixed point (Nash equilibrium), the statistical behaviors of the models under exogenous shocks were examined. Based on the analytical and empirical results obtained, a fundamental **"Stability-Efficiency Trade-off"** was identified for oligopolistic markets:

1. **Stability:** Stochastic variance analysis has shown that although the Cournot model inherently contains simultaneous uncertainty, it is structurally more stable in absorbing exogenous economic shocks (Variance: 2.22). In the Stackelberg model, the first-mover advantage of the leader firm renders the system extremely sensitive and fragile against crises (Variance: 5.00).

2. **Efficiency:** Examining total welfare calculations, it was determined that the aggressive nature of the Stackelberg model increases output and reduces prices, thereby maximizing consumer surplus and generating a higher market welfare (3802.79), a finding that aligns strictly with the established theoretical welfare rankings (Vives,1999). The welfare of the Cournot model remained relatively lower (3604.02).

In conclusion, while regulators encouraging Cournot-style simultaneous competition during crisis periods or in markets with high cost fluctuations preserves system stability, allowing the formation of Stackelberg-style market leadership in stable market conditions will maximize overall economic welfare.

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